The solution(s) to the long-standing issue of the Internal Rate of Return: scientific and educational implications

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Rate of interest and Internal Rate of Return

In the first lesson of a course in financial mathematics…

$$\boldsymbol{P} = (f_0, f_T)$$

⇓

rate of interest

⇓

$$x = \frac{f_T + f_0}{-f_0} = \frac{\text{interest}}{\text{capital}}$$

$$\boldsymbol{P} = (-100{,}110) \qquad x = \frac{110 - 100}{100} = 10\%$$

...after some lessons…

$$\boldsymbol{P} = (f_0, f_1, f_2, \dots, f_T) \in \mathbb{R}^{T+1}$$

$$NPV(r) = \sum_{t=0}^{T} f_t \cdot (1+r)^{-t}$$

$$NPV(x) = \sum_{t=0}^{T} f_t \cdot (1+x)^{-t} = 0$$ *Internal Rate of Return* (IRR)

The IRR problems

(1) multiple real-valued IRRs

(2) complex-valued IRRs

(3) ambiguous meaning (rate of return vs. rate of cost)

(4) incompatibility with the NPV

(5) incompatibility with arbitrage

(6) variable costs of capital

(7) -1 as a possible rate of return

(8) real numbers smaller than -1

(9) no rate of return for arbitrage strategies (?)

(10) no rate of return on the *initial* capital

(11) relations between IRR and accounting rates of return (ROI, ROE)

Definition. *A rate of return for project P is a number fulfilling the following properties:*

(i) *it is a real-valued number*

(ii) *it may be formally represented by a ratio, whose numerator and denominator are interpretable as return and capital, respectively*

(iii) *it is accompanied by another (real-valued) number, which is interpretable as an opportunity cost of capital, and represents the minimum rate of return required by the investors*

(iv) *it supplies a criterion which solves problems (1)-(11) above*

Fundamental relation

$$\overbrace{c_t}^{\text{end-of-period capital}} = \overbrace{c_{t-1}}^{\text{beginning-of-period capital}} + \overbrace{I_t}^{\text{interest}} - \overbrace{f_t}^{\text{cash flow}}$$

Time	capital	Interest	cash flow	rate of interest
0	$c_0 = -f_0$		f_0	
1	c_1	I_1	f_1	$x_1 = I_1/c_0$
2	c_2	I_2	f_2	$x_2 = I_2/c_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$T-1$	c_{T-1}	I_{T-1}	f_{T-1}	$x_{T-1} = I_{T-1}/c_{T-2}$
T	$c_T = 0$	I_T	f_T	$x_T = I_T/c_{T-1}$
	$C := \sum_{t=1}^{T} c_{t-1}$	$I := \sum_{t=1}^{T} I_t$	$f := \sum_{t=0}^{T} f_t$	

$$I = f$$

Replicating portfolio

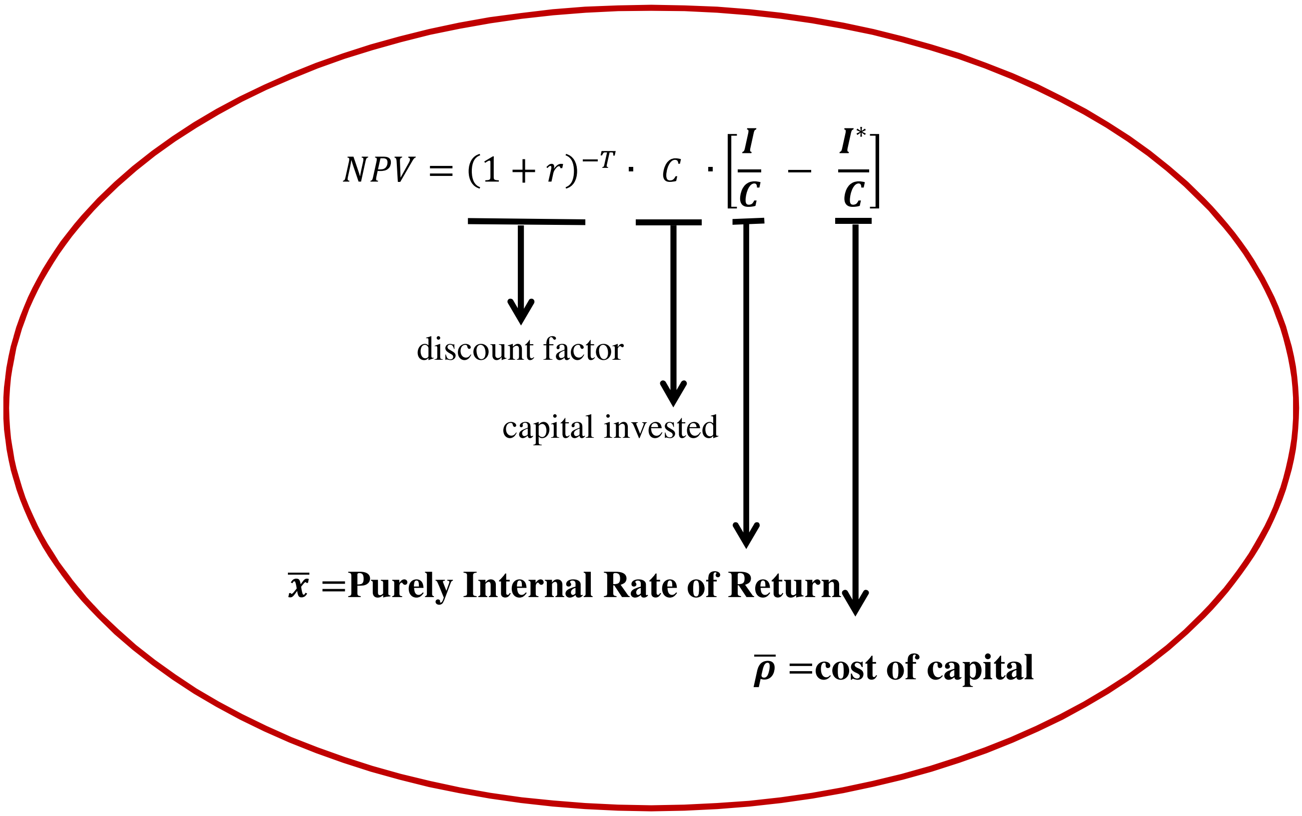
$$\overbrace{c_t^*}^{\text{end-of-period capital}} = \overbrace{c_{t-1}^*}^{\text{beginning-of-period capital}} + \overbrace{r\, c_{t-1}^*}^{\text{interest}} - \overbrace{f_t}^{\text{cash flow}}$$

Time	capital	interest	cash flow	rate of interest
0	$c_0^* = -f_0$		f_0	
1	c_1^*	I_1^*	f_1	$r = I_1^*/c_0^*$
2	c_2^*	I_2^*	f_2	$r = I_2^*/c_1^*$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$T-1$	c_{T-1}^*	I_{T-1}^*	f_{T-1}	$r = I_{T-2}^*/c_{T-2}^*$
T	c_T^*	I_T^*	f_T	$r = I_{T-1}^*/c_{T-1}^*$
	$C^* := \sum_{t=1}^{T} c_{t-1}^*$	$I^* := \sum_{t=1}^{T} I_{t-1}^*$	$f := \sum_{t=1}^{T} f_{t-1}$	

total interest=I

total foregone interest=I^*

$$NPV = (1+r)^{-T} \cdot \overbrace{(I - I^*)}^{NFV}$$



Project acceptability

$$\bar{x} \gtrless \bar{\rho}$$

Which relation do the period rates x_t's bear to the Purely Internal Rate of Return?

PIRR is a weighted average of one-period rates of return

$$\bar{x} = \frac{\boldsymbol{x_1 c_0 + x_2 c_1 + \cdots + x_T c_{T-1}}}{\boldsymbol{c_0 + c_1 + \cdots + c_{T-1}}}$$

Which relation does the (first lesson)'s notion of rate of interest bear to Purely Internal Rate of Return?

$$\bar{x} = \frac{\boldsymbol{I}}{\boldsymbol{C}} = \frac{\boldsymbol{f}}{\boldsymbol{C}} = \frac{\boldsymbol{f_0 + f_1 + \cdots + f_T}}{\boldsymbol{c_0 + c_1 + \cdots + c_{T-1}}}$$

$$\bar{x} = rate\ of\ interest = \frac{\boldsymbol{f_0 + f_T}}{\boldsymbol{-f_0}} = \frac{\boldsymbol{f_0 + f_T}}{\boldsymbol{c_0}}$$

What is $\bar{\rho}$?

$$\boldsymbol{C^{*} := c_{0}^{*} + c_{1}^{*} + c_{2}^{*} \ldots + c_{T-1}^{*}}$$

The investor undertaking the project foregoes to invest C^{*} in the market

$$I^{*} = \bar{\rho} C = r C^{*} = \overbrace{r}^{\text{foregone rate of return}} \cdot C + r \cdot \overbrace{(C^{*} - C)}^{\text{excess capital}}$$

$$\bar{\rho} = \frac{I^{*}}{C} = \overbrace{r}^{\text{foregone rate of return}} + r \frac{\overbrace{C^{*} - C}^{\text{excess capital}}}{C} = r \frac{C^{*}}{C}$$

$\frac{C^{*}}{C}$ serves the purpose of adjusting for capital

The cost of capital is a weighted average as well

$$\bar{\rho} = \frac{\rho_{1} c_{0} + \rho_{2} c_{1} + \cdots + \rho_{T} c_{T-1}}{c_{0} + c_{1} + \cdots + c_{T-1}}$$

$$\rho_{t} := r \frac{c_{t-1}^{*}}{c_{t-1}}$$

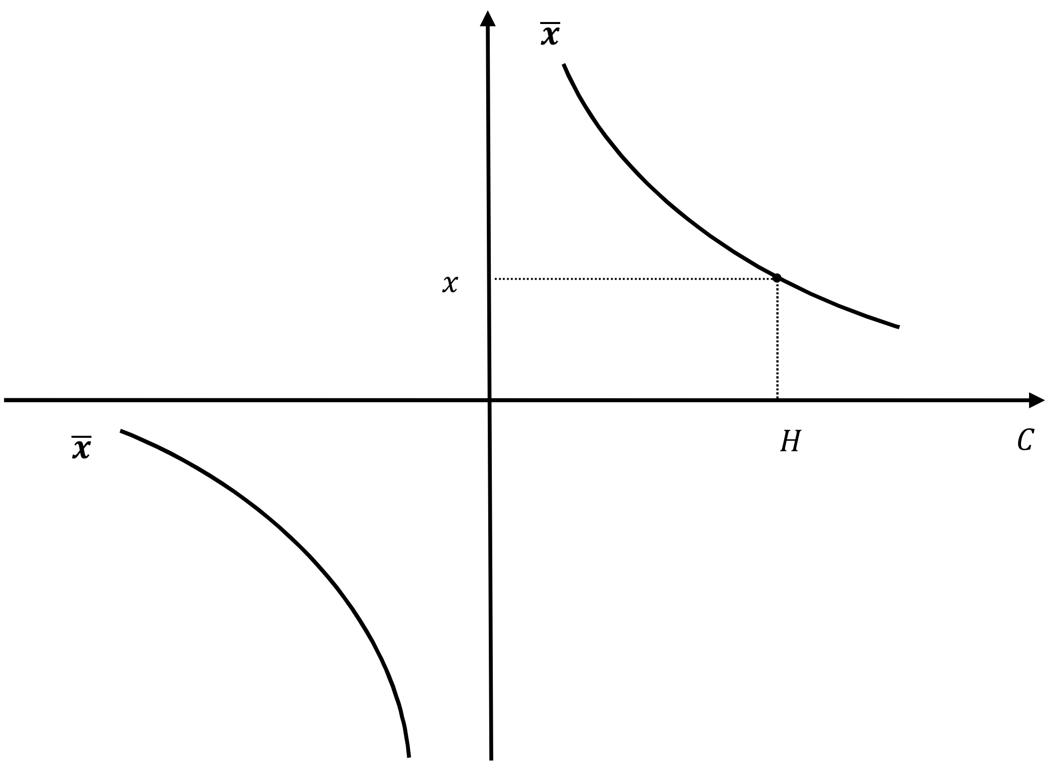
The argument does not depend on depreciation

it holds for any $c=(c_0, c_1, \dots, c_{T-1})$

$$(c_0, c_1, \dots, c_{T-1}) \mapsto C \longmapsto \bar{x}$$

To every project there corresponds a return function, not a return rate: $C \longmapsto \bar{x}$

Purely Internal Return Function



A project is associated with infinitely many pairs $(C, \bar{x})$

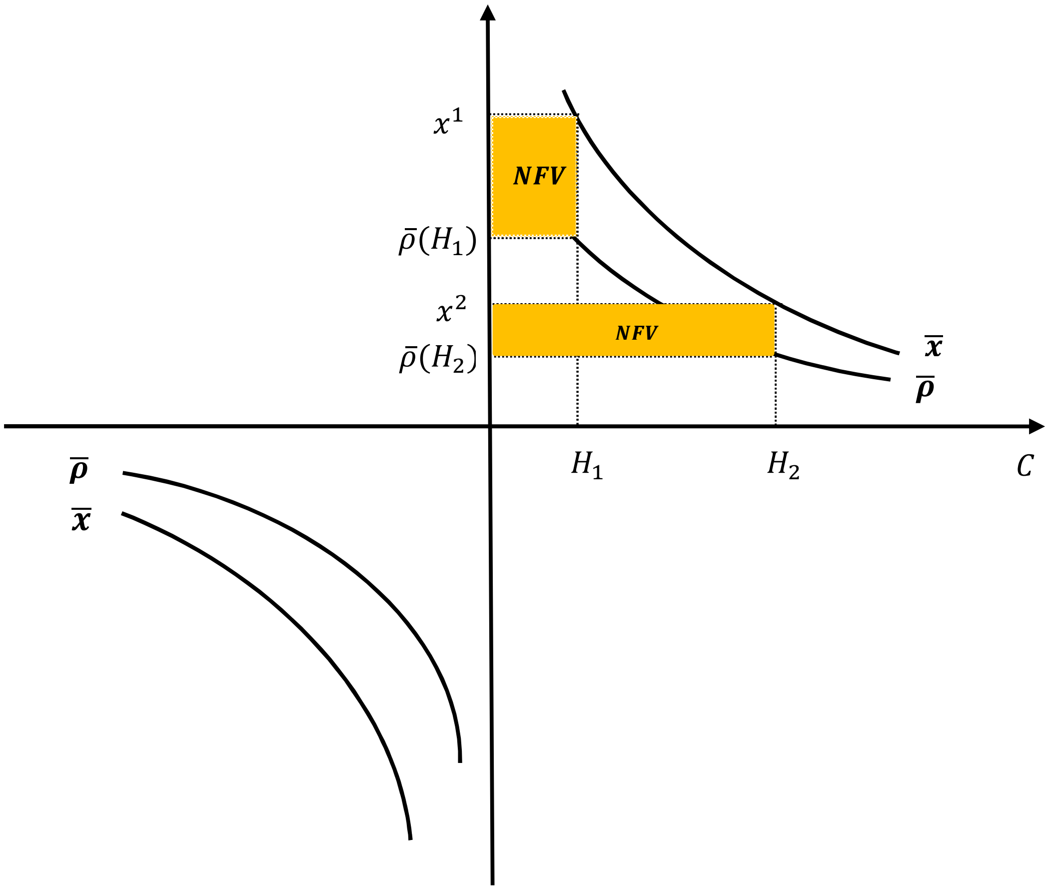
EVERY REAL-VALUED IRR IS A PARTICULAR CASE OF PIRR

$\boldsymbol{H} = \mathbf{1363.6}$

Time	0	1	2	3	4	PIRR
Cash Flows	**−900**	**800**	**100**	**100**	**91**	
Depreciation	c_0 =900	c_1 =226.1	c_2 =157.7	c_3 =79.8	c_4 =0	
		x_1 **=14%**	x_2 **=14%**	x_3 **=14%**	x_4 **=14%**	**14%**
Depreciation	c_0 =900	c_1 =200	c_2 =100	c_3 =163.6	c_4 =0	
		x_1 **=11.11%**	x_2 **=0%**	x_3 **=163.6%**	x_4 **=−44.38%**	**14%**
Depreciation	c_0 =900	c_1 =300	c_2 =123.6	c_3 =40	c_4 =0	
		x_1 **=22.22%**	x_2 **=−25.47%**	x_3 **=13.27%**	x_4 **=127.5%**	**14%**
Depreciation	c_0 =900	c_1 =190	c_2 =157	c_3 =116.6	c_4 =0	
		x_1 **=10%**	x_2 **=35.26%**	x_3 **=37.96%**	x_4 **=−21.96%**	**14%**
Depreciation	c_0 =900	c_1	c_2	$1363.6 - c_1 - c_2$	c_4 =0	
		x_1	x_2	x_3	x_4	**14%**

Every real-valued IRR is just a value taken on by the PIRR function in correspondence of a particular capital H

$$\bar{x}(H) = \frac{\boldsymbol{x_1 c_0 + x_2 c_1 + \cdots + x_T c_{T-1}}}{\boldsymbol{c_0 + c_1 + \cdots + c_{T-1}}} = IRR$$



x^1
NFV
$\bar{\rho}(H_1)$
x^2
NFV
$\bar{\rho}(H_2)$
$\bar{x}$
$\bar{\rho}$
H_1
H_2
C
$\bar{\rho}$
$\bar{x}$

The accounting problem

What is the relation between IRR and accounting rates of return?

Are accounting rates of return economically significant? Can they be used for measuring a project's or a firm's performance?

"it is widely presumed in the accounting and economic literatures that, for the most part in practice, ARRs are artifacts without economic significance." (Peasnell, 1982, p. 368).

Income Statement		Sales		I_t
	−	Cost of Goods Sold		
	−	Depreciation		
	−	General & Administrative Expenses		
	−	Interest		
	=	*Profit Before Taxes*		
	−	Taxes		
	=	**NET INCOME**		
Accruals	+	Depreciation	Δ Fixed Assets	$+(c_{t-1} - c_t)$
	−	New investments		
	−	Increase in operating WC	Δ WC	
	−	Increase in liquid assets		
	−	Interest	Δ Debt	
	+	Increase in debt		
payment to shareholders	=	**EQUITY CASH FLOW**		$= f_t$

$$ROE_t = \frac{\text{Net Income}}{\text{Equity Book Value}} = \frac{I_t}{c_{t-1}}$$

Time	Equity Book value	Net Income	ROE
0	c_0		
1	c_1	I_1	$\frac{I_1}{c_0}$
2	c_2	I_2	$\frac{I_2}{c_1}$
⋮	⋮	⋮	
T	0	I_T	$\frac{I_T}{c_{T-1}}$

$$\bar{x} = \frac{\sum_1^T ROE_t \cdot book\ value_{t-1}}{\sum_1^T book\ value_{t-1}}$$

The ROEs generate the PIRR

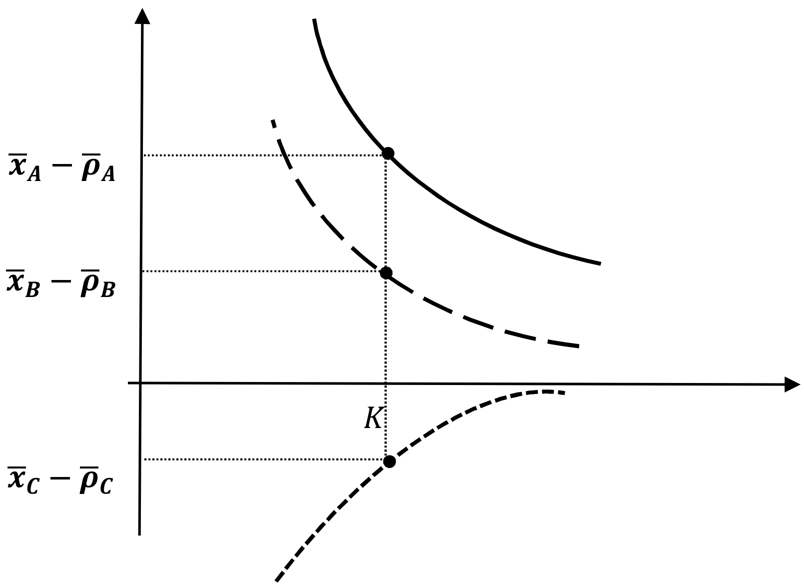
A project (a firm) creates value if and only if the weighted average of accounting rates of return $\bar{x}$ exceeds the cost of capital $\bar{\rho}$, *whatever* the depreciation pattern selected by managers for the fixed assets

Mutually exclusive projects

$$f^{A-B} = f^{A} - f^{B}$$

$$\overline{x}_{A-B} \gtrless \overline{\rho}_{A-B}$$

Ranking projects



Maximize the margin!

Decomposition of NPV (NFV)

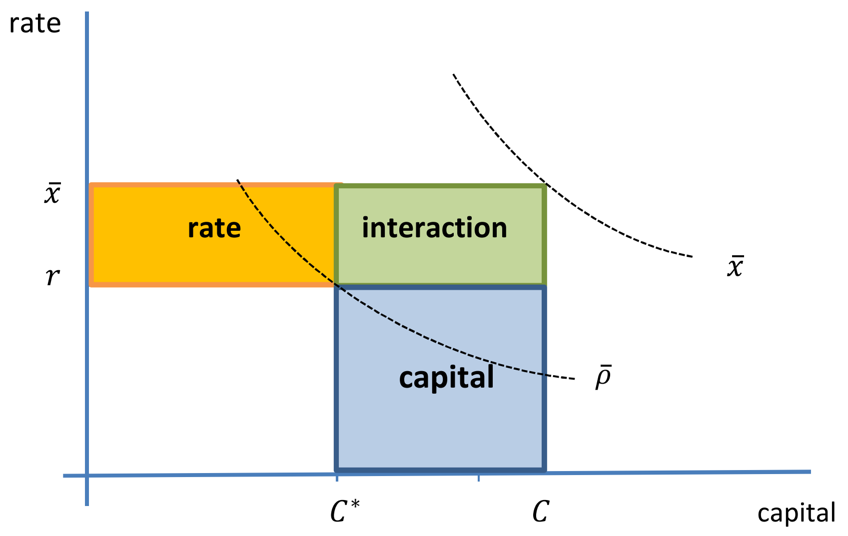
$$\Delta \boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}^1) - \boldsymbol{f}(\boldsymbol{x}^0) = \sum_{i=1}^{n} \Delta_i \boldsymbol{f} + \underbrace{\sum_{i<j}^{n} \Delta_{i,j} \boldsymbol{f} + \cdots + \Delta_{1,2,\ldots,n} \boldsymbol{f}}_{\text{interaction effect}} = \sum_{k=1}^{n} \sum_{i_1,i_2,\ldots,<i_k}^{n} \Delta_{i,j} \boldsymbol{f}$$

$$f(x_1, x_2) = x_1 \cdot x_2$$

$$x^0 = (x_1^0, x_2^0) = (C^*, r) \qquad x^1 = (x_1^1, x_1^1) = (C, \bar{x})$$

$$NFV = C(\bar{x} - \bar{\rho}) = \bar{x}C - rC^* = f(x^1) - f(x^0)$$

$$NFV = \overbrace{C^*(\bar{x} - r)}^{\text{rate}} + \overbrace{r(C - C^*)}^{\text{capital}} + \overbrace{(C - C^*)(\bar{x} - r)}^{\text{interaction effect}}$$



		Source		
		Rate	**Capital**	**Interaction**
Period	1	$c_0^*(x_1 - r)$	**0**	**0**
	2	$c_1^*(x_2 - r)$	$r(c_1^* - c_1)$	$(c_1^* - c_1)(x_2 - r)$
	3	$c_2^*(x_3 - r)$	$r(c_2^* - c_2)$	$(c_2^* - c_2)(x_3 - r)$
		$\vdots$	$\vdots$	$\vdots$
	T	$c_{T-1}^*(x_T - r)$	$r(c_{T-1}^* - c_{T-1})$	$(c_{T-1}^* - c_{T-1})(x_T - r)$

(if required, it is possible to decompose the interaction share into two parts: rate-generated part and capital-generated part)

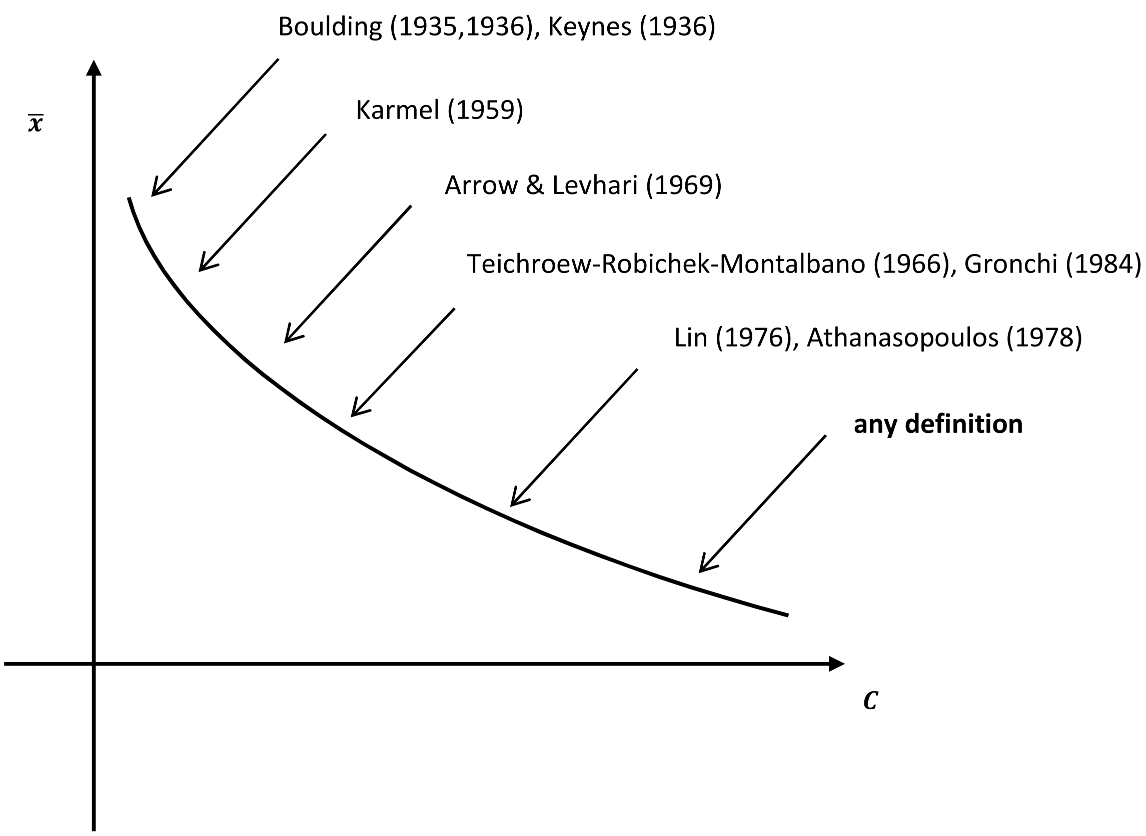
A RATE OF RETURN IS INTEREST ON CAPITAL

CAPITAL IS FORMALLY AND FINANCIALLY ARBITRARY

EVERY PROJECT IS ASSOCIATED WITH A RETURN *FUNCTION*

Any definition of rate of return as a real number is just a particular case of PIRR

This approach incorporates every possible definition of rate of return as a real number



Educational implications:

Course of basic finance and financial mathematics
Courses of accounting and corporate finance

(i) The main financial notions (compounded interest, rate of interest, rate of return, residual income/excess profit, present value, arbitrage) are derived (i.e., the fundamental relation)

(ii) The role of the IRR is diminished

(iii) Conciliation between project rate of return, Net Present Value and rational choice theory

(iv) accounting depreciation is conceptually equivalent to debt's depreciation

(v) income, return and interest are synonyms

(vi) accounting rates of return are economic rates of return

A second solution from Makeham (1874)

$$K_r := \sum_{t=1}^{T} \overbrace{K_t}^{\text{principal repayment}} (1+r)^{-t}$$

$$K_t = c_{t-1} - c_t$$

Makeham's formula

$$\text{Value} = K_r + \frac{IRR}{r}(c_0 - K_r)$$

Generalization

$$\text{Value} = K_r + \frac{\bar{x}_r}{r}(c_0 - K_r)$$

$$\bar{x}_r = \frac{\sum_1^T x_t c_{t-1}(1+r)^{-t}}{\sum_1^T c_{t-1}(1+r)^{-t}} = \frac{\sum_1^T x_t c_{t-1}(1+r)^{-t}}{C_r}$$

Average Internal Rate of Return (AIRR)

$$NPV = \overbrace{(c_0 - K_r)}^{\text{capital sacrificed}} \cdot \left(\frac{\bar{x}_r}{r} - 1\right)$$

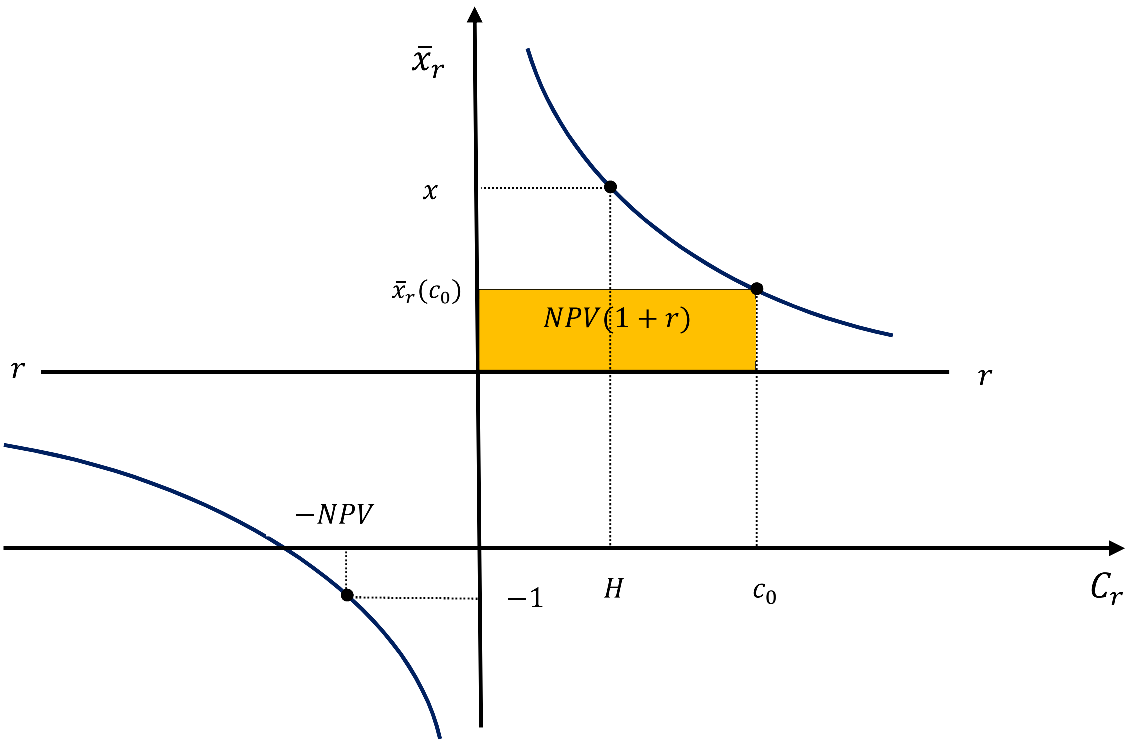
Project acceptability

$\bar{x}_r \gtrless r$

$$NPV = \frac{c_0 - K_r}{r}(\bar{x}_r - r) \quad \Rightarrow \quad \bar{x}_r = r + \frac{r \cdot NPV}{c_0 - K_r}$$

$\Updownarrow$

$$NPV = \frac{C_r}{1+r}(\bar{x}_r - r) \quad \Rightarrow \quad \bar{x}_r = r + \frac{(1+r) \cdot NPV}{C_r}$$



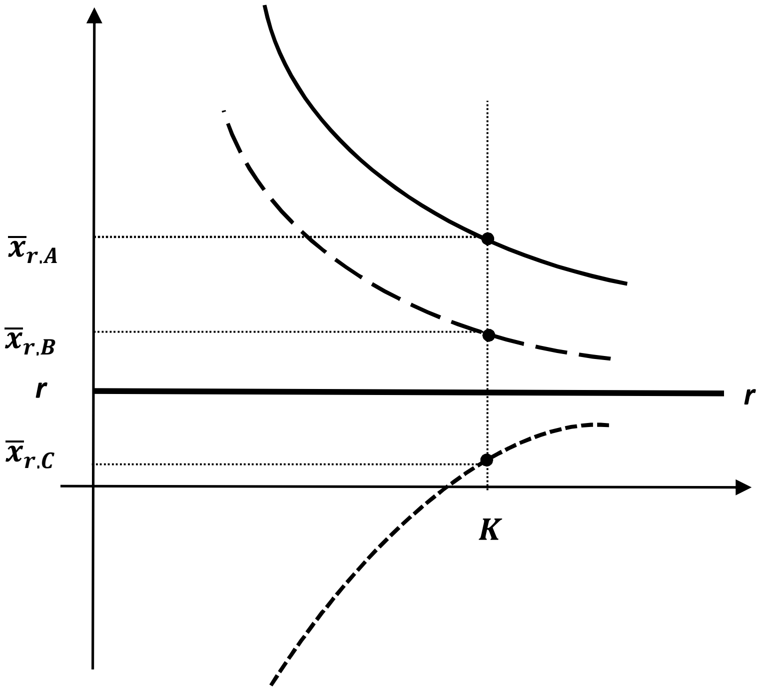
An IRR is a particular case of AIRR

Mutually exclusive projects

$$f^{A-B} = f^{A} - f^{B}$$

$$\bar{x}_{r,A-B} \gtrless r$$

Project ranking



Maximize the AIRR

Arbitrage

0	1	...	T
$A > 0$	0	0	0

Pick $C = c_0 = -A$

$$PIRR = \frac{\text{algebraic sum of cash flows}}{\text{capital invested}} = \frac{A}{-A} = -1 \Longrightarrow \text{it is a rate of cost}$$

$$COC = r\frac{C^*}{C} = r\frac{A\left(\frac{(1+r)^{-T}-1}{r}\right)}{A} = (1+r)^{-T} - 1$$

$$NPV(r) = \frac{-A(-1-[(1+r)^{-T}-1])}{(1+r)^T} = A$$

Pick $C_r = c_0 = -A \quad (\text{e.g. } c_t = 0\ t > 0)$

$$AIRR = \frac{I_1 + 0 + \cdots + 0}{-A} = \frac{A}{-A} = -1$$

$$NPV(r) = -A(-1-r)(1+r)^{-1} = A$$

Any project may be seen as an arbitrage

$$\boldsymbol{P} = (f_0, f_1, f_2, \dots, f_T) \in \mathbb{R}^{T+1}$$

$$NPV(r) > 0$$

Fix c_t so that $C = -\sum_0^T f_t = -NPV(0)$. This implies

$$PIRR = \frac{\sum_0^T I_t}{-\sum_0^T f_t} = \frac{\sum_0^T f_t}{-\sum_0^T f_t} = -1$$

$$NPV(r) = -NPV(0)(-1 - \bar{\rho})(1+r)^{-T}$$

$$= -NPV(0)\left(-1 - r\frac{C^*}{C}\right)(1+r)^{-T} = -NPV(0)\left(-1 - r\frac{C^*}{-NPV(0)}\right)(1+r)^{-T} =$$

$$(NPV(0) - rC^*)(1+r)^{-T} = \left(\sum_{t=1}^{T} I_t - rC^*\right)(1+r)^{-T}$$

Fix c_t so that $C_r = -NPV(r)$

$$AIRR = \frac{NPV(r)}{-NPV(r)} = -1$$

$$NPV(r) = -NPV(r)(-1 - r)(1+r)^{-1}$$

$\overline{x}_r$ may be seen as a traditional IRR in many ways:

First interpretation: one-period project

Time	0	1
Cash flows	$-C_r$	$C_r(1+\bar{x}_r)$

$$NPV(y) = -C_r + \frac{C_r(1+\overline{x}_r)}{1+y} = 0 \Leftrightarrow y = \overline{x}_r$$

Second interpretation: market-determined rate of return

Time	0	1
Cash flows	$-c_0$	$V_1 + f_1$

$$V_1 = \sum_{t=2}^{T} \frac{f_t}{(1+r)^{t-1}}$$

$$\bar{x}_r = \frac{V_1 + f_1 - c_0}{c_0}$$

Third interpretation: perpetual coupon bond

Time	0	1	2	3	⋯
Cash flows	$-[c_0 - K_r]$	$\bar{x}_r[c_0 - K_r]$	$\bar{x}_r[c_0 - K_r]$	$\bar{x}_r[c_0 - K_r]$	⋯

$c_0 - K_r = \text{face value} \qquad \bar{x}_r = \text{coupon rate}$

$$NPV(y) = -(c_0 - K_r) + \sum_{t=1}^{\infty} \frac{\bar{x}_r[c_0 - K_r]}{(1+y)^t}$$

Fourth interpretation: same capital, same residual income

Time	0	1	2	⋯	T
Cash flows	$-c_0$	$\bar{x}_r(c_0) \cdot c_0$	$\bar{x}_r(c_0) \cdot c_0$	⋯	$c_0(1 + \bar{x}_r(c_0))$

$c_t = c_0$ for all $t < T$ equal to the capitals of project P

Constant residual income $c_0(\bar{x}_r(c_0) - r)$ equal to the average residual income of P

$\text{NPV} = c_0(\bar{x}_r(c_0) - r)$ which is equal to the NPV of P

…and infinitely many others

PIRR and AIRR contrasted		
	PIRR	**AIRR**
Definition (return measure)	$\bar{x} := \frac{\sum_1^T I_t}{C}$ $= \frac{\sum_1^T x_t c_{t-1}}{\sum_1^T c_{t-1}}$	$\bar{x}_r := \frac{\sum_1^T I_t (1+r)^{-(t-1)}}{C_r}$ $= \frac{\sum_1^T x_t c_{t-1} (1+r)^{-(t-1)}}{\sum_1^T c_{t-1} (1+r)^{-(t-1)}}$
Definition (cash-flow measure)	$\bar{x} := \frac{\sum_0^T f_t}{C}$	
dependence on the market rate	NO	YES
Definition of investment	$C > 0$	$C_r > 0$
dependence on the market rate	NO	YES
Hurdle rate	$\bar{\rho} := r \cdot \left(\frac{C^*}{C}\right)$	r
dependence on the market rate	YES	YES
Residual rate of return	$\bar{x} - \bar{\rho}$	$\bar{x}_r - r$
Criterion of project acceptability and choice between mutually exclusive projects	$\bar{x} > \bar{\rho}$	$\bar{x}_r > r$
Criterion of project ranking	$\max\ (\bar{x} - \bar{\rho})$	$\max\ (\bar{x}_r - r) = \max\ \bar{x}_r$
Hurdle rate with variable market rates	$\bar{\rho} := \frac{\sum_1^T r_t c_{t-1}^*}{C}$	$\bar{r} := \frac{\sum_1^T r_t c_{t-1}}{C}$
Net Present Value	$C(\bar{x} - \bar{\rho})(1+r)^{-T}$	$C_r(\bar{x}_r - r)(1+r)^{-1}$

Related papers

Magni C.A. (2010). Average Internal Rate of Return and investment decisions: A new perspective. ***The Engineering Economist***, 55(2), 150–180.

Magni C.A. (2010). Purely Internal Rate of Return and investment decisions: a cash-flow perspective. <http://ssrn.com/abstract=1649604>.

Magni C.A (2010). Residual income and value creation: An Investigation into the Lost-Capital Paradigm . ***European Journal of Operational Research***, 201(2), 505–519.

Magni C.A. (2009). Accounting and economic measures: An integrated theory of capital budgeting. Revised version in progress. <http://ssrn.com/abstract=1498106>

Magni C.A. (2009). Splitting up value: A critical review of residual income theories. ***European Journal of Operational Research***, 198(1) (October), 1–22.